Introduction to Eigenvalues and Eigenvectors

part 1

Definition:

Let A be an $n \times n$ matrix. A scalar λ is called an eigenvalue of A if there is a non zero vector $\overline{x}$ such that $A\vec{x} = \lambda\vec{x}$. Such a vector $\vec{x}$ is called an eigenvector of A corresponding to λ.

Show that $\overline{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$ corresponding to $\lambda = 4$

$$A\overline{x} \stackrel{?}{=} \lambda\overline{x}$$

$$\begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

the vector with the entries $\Rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigen vector of the matrix A corresponding to the eigenvalue of 4.

$$\begin{bmatrix} 3 \cdot 2 + 2 \cdot 1 \\ 3 \cdot 2 + (-2) \cdot 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \Longrightarrow \begin{bmatrix} 8 \\ 4 \end{bmatrix} \stackrel{\checkmark}{=} \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

Note: If λ is an eigen value of A, and $\overline{x}$ is an eigenvector belonging to λ, any non zero multiple of $\overline{x}$ will be an eigenvector.

Assuming: $A\vec{x} = \lambda\vec{x}$

$A(\alpha\overline{x}) = \alpha A\overline{x} = \alpha\lambda\vec{x} = \lambda(\alpha\overline{x})$

the process:

To solve for the eigenvalues, λ_i, and the corresponding eigenvectors, $\overline{x}_i$, of an $n \times n$ matrix A, do the following:

① multiply an $n \times n$ identity matrix by the scalar

② Subtract the identity matrix multiple from the matrix A.

③ Find the determinant of the matrix and the difference

④ Solve for the values of λ that satisfy the equation $\det(A - \lambda I) = \vec{0}$.

⑤ Solve for the corresponding vector to each λ.

Find the eigen values and eigen vectors of the matrix:

$$A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$$

① $\lambda I = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

② $A - \lambda I = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$= \begin{bmatrix} 7-\lambda & 3 \\ 3 & -1-\lambda \end{bmatrix}$

③ $\det = \begin{bmatrix} 7-\lambda & 3 \\ 3 & -1-\lambda \end{bmatrix}$

$= (7-\lambda)(-1-\lambda) - (3)(3)$

$= -7 - 7\lambda + \lambda + \lambda^2 - 9$

$= \lambda^2 - 6\lambda - 16$

④ $\lambda^2 - 6\lambda - 16 = 0$

solve for λ

$(\lambda - 8)(\lambda + 2) = 0$

$\lambda - 8 = 0$ $\qquad$ $\lambda + 2 = 0$

$+8 \ \ +8$ $\qquad$ $-2 \ \ -2$

$\lambda = 8$ $\qquad$ $\lambda = -2$

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④ $\lambda^2 - 6\lambda - 16 = 0$

solve for λ

$(\lambda - 8)(\lambda + 2) = 0$

$$\begin{array}{rl} \lambda - 8 &= 0 \\ +8 & +8 \\ \hline \lambda &= 8 \end{array} \qquad \begin{array}{rl} \lambda + 2 &= 0 \\ -2 & -2 \\ \hline \lambda &= -2 \end{array}$$

So lambda $\lambda = 8, -2$

So the eigen values are $\lambda = 8$, $\lambda = -2$

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$$\begin{bmatrix} 7-\lambda & 3 \\ 3 & -1-\lambda \end{bmatrix}$$

$\lambda = 8$

$$\begin{bmatrix} 7-8 & 3 \\ 3 & -1-8 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}}_{B}$$

Solve: $B\overline{X} = 0$

$$\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

now Perform Row Reduction

$-1R_1 \rightarrow R_1$

$$\left[\begin{array}{cc|c} 1 & -3 & 0 \\ 3 & -9 & 0 \end{array}\right]$$

now $-3R_1 + R_2 \rightarrow R_2$

$$\begin{array}{cc} x_1 & x_2 \end{array}$$
$$\left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 - 3x_2 = 0$

$x_1 = 3x_2$

x_2 is arbitrary

Let $x_2 = 1$

the eigen vector is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$X_1 = 3(1)$

$X_1 = 3$

so $X_1 = 3$

$X_2 = 1$

so the $\lambda = 8$: $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\lambda = -2$

$$\begin{bmatrix} 7-(-2) & 3 \\ 3 & -1-(-2) \end{bmatrix} = \underbrace{\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}}_{C}$$

solve : $C\vec{X} = 0$

$$\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

now perform Row Reduction

$\frac{R_1}{9} \to R_1$

$$\left[\begin{array}{cc|c} 9 & 3 & 0 \\ 3 & 1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 3 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 3 & 1 & 0 \end{array}\right]$$

$-3R_1 + R_2 \to R_2$

$$\left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 + \frac{1}{3}x_2 = 0$

$x_1 = -\frac{1}{3}x_2$

x_2 is arbitrary

let $x_2 = -3$

so: $x_1 = -\frac{1}{3}(-3)$

$x_1 = 1$

so: $x_1 = 1$

$x_2 = -3$

$\lambda_2 = -2 : \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$\lambda_1 = 8 : \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \lambda_2 = -2 : \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

Check: $\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 8 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 24 \\ 8 \end{bmatrix} = \begin{bmatrix} 24 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -3 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$