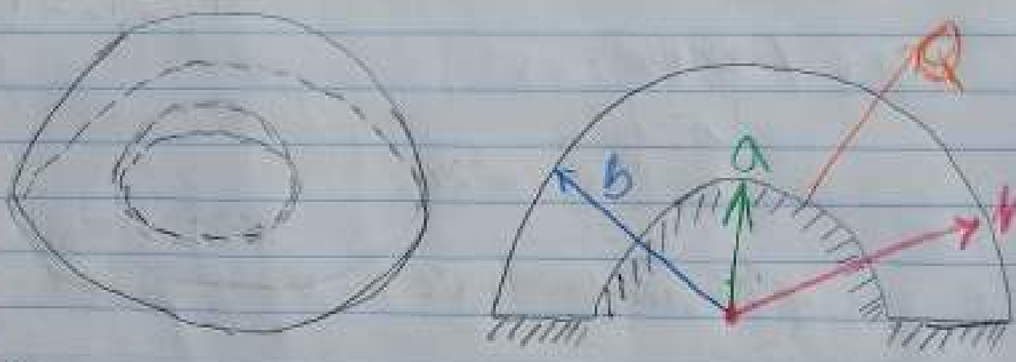
Test 2

Problem:

Purpose: To develop an expression for the one-dimensional steady-state temperature distribution $T(r)$ within the hemisphere. To develop the expression for heat transfer at $r=b$. To assume C_1 has units of temperature per unit length and C_2 has units of temperature. To check the units for the heat transfer equation. To check the results of the temperature profile and heat transfer equations when $g=0$ and if it makes sense.

Drawing:

• h • T_∞



Sources:

- Test 2
- Bayazitoglu Y., Ozisik N., "A Textbook for Heat Transfer Fundamentals" Begell House Inc. (2012)

Design Considerations

- Steady-state
- one-dimensional heat transfer
- heat generation

Data and Variables

- Inner radius $= a$
- Outer radius $= b$
- $g = g_0 \frac{b^2}{r^2} \left(\frac{W}{m^3}\right)$
- $k = k_0 \frac{b^2}{r^2} \left(\frac{W}{m \cdot K}\right)$
- $\frac{1}{r} \frac{d}{dr}\left(r^2 k \frac{dT}{dr}\right) + g = 0$
- $g_0 =$ constant
- $k_0 =$ constant

Procedure:

- I am going to use the differential equation for a hemisphere and when the thermal conductivity depends on the radius to develop the expression for the temperature distribution:

$$\frac{1}{r^2} \frac{d}{dr}\left(r^2 k \frac{dT}{dr}\right) + g = 0.$$

- I will solve that expression with the equations for heat generation and conductivity that depend on the radius:

$$g = g_0 \frac{b^2}{r^2} \qquad k = k_0 \frac{b^2}{r^2}$$

- I will solve that expression between the two boundary conditions. Since the inside surface of the sphere is insulated, the first boundary condition is:

$$r = a \qquad \frac{dT}{dr} = 0$$

The outside of the hemisphere is subjected to dissipation of heat by convection, the second boundary condition is:

$$r = b \qquad -k \left.\frac{dT}{dr}\right|_{r=b} = h\left(T|_{r=b} - T_\infty\right)$$

- Once I develop the temperature profile equation, I will use equation for convective heat transfer and that temperature profile equation to develop an equation for the heat transfer at $r = b$.

$$Q = hA(T - T_\infty)$$

Calculations

- Differential equation when thermal conductivity depends on radius:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 k\frac{dT}{dr}\right)+g=0$$

- Conductivity changes with radius following this equation:

$$k=k_o\frac{b^2}{r^2}$$

- heat generation changes with radius following this equation:

$$g=g_o\frac{b^2}{r^2}$$

Boundary Conditions

$r=a$

$$\frac{dT}{dr}=0$$

$r=b$

$$-k\left.\frac{dT}{dr}\right|_{r=b}=h\left(T_{r=b}-T_\infty\right)$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\cancel{r^2}k_o\frac{b^2}{\cancel{r^2}}\frac{dT}{dr}\right)+g_o\frac{b^2}{r^2}=0$$

$$\left[\frac{k_o b^2}{r^2}\frac{d^2T}{dr^2}+g_o\frac{b^2}{r^2}=0\right]\times\frac{r^2}{k_o b^2}$$

$$\frac{d^2T}{dr^2}+\frac{g_o}{k_o}=0$$

$r=a$ $\quad$ $r=b$

$$\frac{dT}{dr}=0 \qquad -k\left.\frac{dT}{dr}\right|=h\left(T-T_\infty\right)$$

$$\left[\frac{d}{dr}\left(\frac{dT}{dr}\right)+\frac{g_o}{k_o}=0\right]\times dr$$

$$\int d\left(\frac{dT}{dr}\right)+\int\frac{g_o}{k_o}\,dr=\int 0\,dr$$

$$\frac{dT}{dr}+\frac{g_o r}{k_o}=C_1$$

- Check to see if C_1 has units of temperature per unit length.

$$\frac{K}{m} + \frac{\frac{W}{m^3}}{\frac{W}{m \cdot K}} \cdot m = C_1$$

$$\frac{K}{m} + \cancel{\frac{W}{m^3}} \cdot \frac{m \cdot K}{\cancel{W}} \cdot m = C_1$$

$$\frac{K}{m} = C_1$$

- C_1 does have units of temperature per unit length

$$\left[\frac{dT}{dr} + \frac{g_0}{K_0} r = C_1\right] \times dr$$

$$\int dT + \int \frac{g_0}{K_0} r\,dr = \int C_1\,dr + \int 0\,dr$$

$$T + \frac{g_0}{K_0}\frac{r^2}{2} = C_1 r + C_2$$

$$\boxed{T = -\frac{g_0}{K_0}\frac{r^2}{2} + C_1 r + C_2}$$

$r = a \quad \frac{dT}{dr} = 0$

$$\frac{dT}{dr} + \frac{g_0}{K_0} r = C_1$$

$$0 + \frac{g_0}{K_0} a = C_1$$

$$C_1 = \frac{g_0}{K_0} a$$

$$r = b \quad -K \left.\frac{dT}{dr}\right|_{r=b} = h\left(T_1 - T_\infty\right)\Big|_{r=b}$$

$$-K_0 \frac{b^2}{r^2}\left[C_1 - \frac{g_0}{K_0} r\right]\Big|_{r=b} = h\left[-\frac{g_0}{K_0}\frac{r^2}{2} + C_1 r + C_2 - T_\infty\right]\Big|_{r=b}$$

$$-K_0 \cancel{\frac{b^2}{b^2}}\left[\frac{g_0}{K_0} a - \frac{g_0}{K_0} b\right] = h\left[-\frac{g_0}{K_0}\frac{b^2}{2} + \frac{g_0}{K_0} a b + C_2 - T_\infty\right]$$

$$-\cancel{K_0}\frac{g_0}{\cancel{K_0}}(a-b) = h\left[\frac{g_0}{K_0} b\left(a - \frac{b}{2}\right) + C_2 - T_\infty\right]$$

$$-\frac{g_0}{h}(a-b) = \frac{g_0}{K_0} b\left(a - \frac{b}{2}\right) + C_2 - T_\infty$$

$$C_2 = -\frac{g_0}{h}(a-b) - \frac{g_0}{K_0} b\left(a - \frac{b}{2}\right) + T_\infty$$

- Check to see if C_2 has units of temperature

$$C_2 = -\frac{\frac{W}{m^3}}{\frac{W}{m^2 \cdot K}}(m - m) - \frac{\frac{W}{m^3}}{\frac{W}{m \cdot K}} m\left(m - \frac{m}{2}\right) + K\infty$$

$$C_2 = \cancel{\frac{W}{m^3}} \cdot \frac{\cancel{m^2} \cdot K}{\cancel{W}} \cdot m - \frac{W}{m^3} \cdot \frac{m \cdot K}{W} \cdot m^2 + K$$

$$C_2 = K$$

- C_2 does have units of temperature

Next Page

$$T = -\frac{g_0}{K_0}\frac{r^2}{2} + C_1 r + C_2$$

$$\boxed{T = -\frac{g_0}{K_0}\frac{r^2}{2} + \frac{g_0}{K_0} a r + \frac{g_0}{h}(a-b) - \frac{g_0}{K_0} b\left(a - \frac{b}{2}\right) + T_\infty}$$

$$T = -\frac{g_0}{2K_0} r^2 + \frac{g_0}{K_0} a r - \frac{g_0}{h}(a-b) - \frac{g_0}{K_0} ba + \frac{g_0}{2K_0} b^2 + T_\infty$$

$$\boxed{T = -\frac{g_0}{2K_0}(r^2 - b^2) + \frac{g_0}{K_0} a (r-b) - \frac{g_0}{h}(a-b) + T_\infty}$$

Now, I need to develop the equation for heat transfer at $r = b$

Heat transfer due to convection

$$Q = hA(T - T_\infty)$$

$$T - T_\infty = -\frac{g_0}{2K_0}(r^2 - b^2) + \frac{g_0}{K_0} a (r-b) - \frac{g_0}{h}(a-b)$$

$$Q = hA\left[\frac{-g}{2K_0}(r^2 - b^2) + \frac{g_0}{K_0} a (r-b) - \frac{g_0}{h}(a-b)\right]$$

Solving for Q at $r = b$

$$Q = hA\left[\frac{-g}{2K_0}(\underbrace{\cancel{b^2 - b^2}}_{0}) + \frac{g_0}{K_0} a (\underbrace{\cancel{b-b}}_{0}) - \frac{g_0}{h}(a-b)\right]$$

$$Q = hA\left(-\frac{g_0}{h}(a-b)\right)$$

$$\boxed{Q = -g_0 A (a-b)}$$

I will check the units of the heat transfer equation

$$Q = -\frac{W}{\cancel{m^2}} \cdot \cancel{m^2} \left(\cancel{\frac{m}{m}}\right)$$

$$Q = W$$

Equation for heat transfer does have the correct units.

Summary:

- The temperature profile for this hemisphere when thermal conductivity and heat generation depend on radius is:

$$T = -\frac{g_0}{2k_0}\left(r^2 - b^2\right) + \frac{g_0}{k_0} a (r-b) - \frac{g_0}{h}(a-b) + T_\infty$$

- While developing the temperature profile equation, I checked the units of C_1 to ensure it is temperature per unit length. I checked the units of C_2 to ensure it had units of temperature.

The equation for heat transfer at $r = b$ is:

$$Q = -g_0 A (a-b)$$

- I checked the units of the heat transfer equation to ensure it had units of watts.

Materials

- hemispherical fuel element whose thermal conductivity depends on radius following the expression $k = k_0 (b^2/r^2)$

Analysis

- Since the inside and bottom of this hemispherical fuel element are insulated, the outside is subjected to convective heat transfer, and heat is being generated within it, the temperature will be the greatest at $r=a$ and the temperature will decrease as r increases.

If g_0 was zero and we are looking at the temperature profile equation, we would get:

$$T = -\frac{0}{2k_0}(r^2-b^2) + \frac{0}{k_0}a(r-b) - \frac{0}{h}(a-b) + T_\infty \rightarrow T = T_\infty$$

T would equal T_∞. This makes sense because if there is no heat being generated within the fuel element, after a long time, it would reach steady state and the temperature of the element would be T_∞. Looking at the heat transfer equation, if g_0 was zero, we would get:

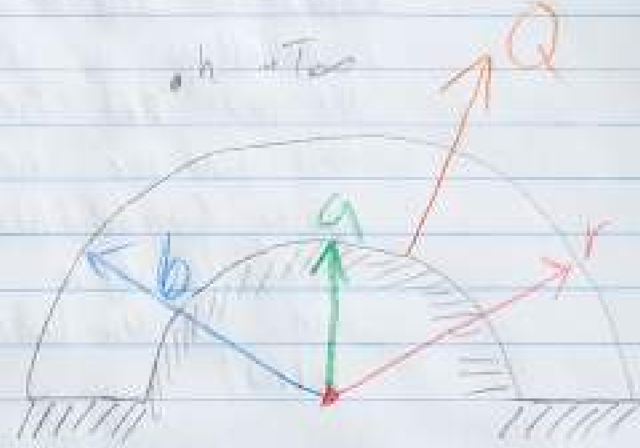
$$Q = -0 \cdot A(a-b) \longrightarrow Q = 0$$

Q would equal zero. This makes sense for the same reason in the temperature profile equation. If $g=0$, then $T = T_\infty$. This means ΔT is zero, and Q must be zero.

Problem 2:

Purpose: To compute the hole temperature for problem 1 using Comsol. To compare the analytical and numerical temperature profiles. To compare the heat transfer leaving at r=b and calculate the percent error

Drawings:



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Design Considerations
- Steady - state
- one - dimensional heat transfer
- heat generation

Data and Variables
- $k_0 = 10\ W/m\cdot K$
- $h = 200\ W/m^2\cdot K$
- $a = 2cm = 0.02m$
- $b = 5cm = 0.05m$
- $T_\infty = 100°C$
- $g_0 = 10^6\ W/m^3$

· Procedure

- I will solve the 1st problem using comsol and the variables provided for this question. After I get the node temperatures, I will use that data to create a numerical temperature profile. Next, I will enter the analytical temperature profile equation that I developed in problem 1. I will compare the the two and see if they match. Next, I will find the heat transfer at $r=b$ using Comsol and compare that to the results from the expression I derived in problem 1 and calculate the percent error.

Calculations:

a) I solved the first problem using the variables provided in problem 2 and computed the node temperatures

- Note: in this problem, since thermal conductivity and heat generation change with temperature, I must enter in the expressions:

$$g = g_0 \frac{b^2}{r^2}$$

$$g = g_0 \cdot b^2 / r^2$$

- Since Comsol using Cartesian coordinates, r must be written as:

$$r = \sqrt{x^2 + y^2 + z^2}$$

- The expression becomes:

$$g = g_0 \cdot b^2 / \left(\sqrt{x^2+y^2+z^2}\right)^2$$

1

- The same case applies to thermal conductivity

$$K = K_0/b^2 \left(\sqrt{x^2+y^2+z^2}\right)^2$$

b) After calculating the node temperatures in Comsol, I developed the numerical temperature profile using a line graph with the data set from a Cut Line 3D.

I created a second line graph and entered the temperature profile expression developed in problem 1.

$$T = -\frac{g}{2K_0}\left(r^2-b^2\right) + \frac{g_0}{K_0}a(r-b) - \frac{g_0}{h}(a-b) + T_\infty$$

- Now, I have the analytical temperature profile.
- I compared the analytical and numerical temperature profiles and they are exactly the same as they should be.

c) Using Comsol, I computed the heat transfer at the surface at $r=b$. I found the heat transfer to be 471.53 W.

- I will calculate heat transfer at $r=b$ using the expression derived in problem 1.

$$Q = -g_0 A (a-b)$$

- The area in this case is the outside of the hemisphere

- The area of sphere is

$$A = 4\pi r^2$$

- The rounded area of the hemisphere will be 1/2 the area of the sphere

$$A = \frac{4\pi r^2}{2}$$

$$A = 2\pi r^2$$

$$Q = -g_0 \cdot 2\pi r^2 (a-b)$$

Solving for Q at $r=b$

$$Q = -g_0 \cdot 2\pi b^2 (a-b)$$

$$Q = -10^6 \frac{W}{m^3} \cdot 2 \cdot \pi \cdot 0.05m^2 (0.02m - 0.05m)$$

$$Q = 471.24 \, W$$

The heat transfer at $r=b$ from COMSOL is 471.53 W. The heat transfer at $r=b$ using the expression derived from problem 1 is 471.24 W. The results are very similar.

$$\text{Percentage error} = \left| \frac{\text{numerical} - \text{analytical}}{\text{analytical}} \right| \times 100\%$$

Percent error = 0.062%

Summary:

The node temperatures ranges from 295°C to 250°C. The numerical and analytical temperature profiles were an exact match as they should be. The heat transfer calculated using the expression I developed in problem 1 is 471.24 W. The heat transfer I calculated using Comsol is 471.53 W. The percent error is 0.062%.

Materials:

- Hemispherical fuel element whose conductivity depends on radius following this equation:

$$K = K_0 (b^2/r^2)$$

- where $K_0 = 10\ W/m\cdot K$

• Analysis.

The node temperatures decrease as r increases with the highest temperature at r=a as I said in my analysis on problem 1. The heat transfer I calculated using Comsol is slightly higher than the heat transfer I calculated using the expression I developed in problem 1. However after calculating the percent error to be 0.062%. It is much less than 1% which means that the difference between them can be considered as negligible.